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# Horizon Instability in the Cross Polarized Bell - Szekeres Spacetime

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**Abstract**

The quasiregular singularities (horizons) that form in the collision of cross polarized electromagnetic waves are, as in the linear polarized case, unstable. The validity of the Helliwell-Konkowski stability conjecture is tested for a number of exact backreaction cases. In the test electromagnetic case the conjecture fails to predict the correct nature of the singularity while in the scalar field and in the null dust cases the aggrement is justified.

# 1 Introduction

It has been known for a long time that owing to planar property and mutual focussing, colliding plane waves (CPW) result in spacelike singularities [1]. These singularities are somewhat weakened when the waves are endowed with a relative cross polarization prior to the collision. A solution given by Chandrasekhar and Xanthopoulos (CX) [2], however constitutes an example contrasting this category, namely, it possesses a Cauchy Horizon (CH) instead of a spacelike singularity. Naturally, this solution initiated a literature devoted entirely on the quest of stability of horizons formed hitherto. CH formed in spacetimes of CPW was shown by Yurtsever to be unstable against plane-symmetric perturbations [3]. A linear perturbation analysis by CX reveals also an analogues result [4]. Any such perturbation applied to a CPW spacetime will turn the CH into an essential singularity.

A second factor that proved effective in weakening the strength of a singularity in CPW is the electromagnetic (em) field itself. In other words, the degree of divergence in the curvature scalars of colliding pure gravitational waves turn out to be stronger than the case when em field is coupled to gravity. In particular, collision of pure em waves must have a special significance as far as singularity formation is concerned. Such an interesting solution was given by Bell and Szekeres (BS) which describes the collision of two linearly polarized step em waves [5]. The singularity (in fact a CH) formed in the interaction region of the BS solution was shown to be removable by a coordinate transformation. On the null boundaries, however it possesses esential curvature singularities which can not be removed by any means. Since cross polarization and em field both play roles in the nature of resulting singularity it is worthwhile to purse these features together. This invokes a cross polarized version of the BS (CPBS) solution which was found long time ago [6,7]. This metric had the nice feature that the Weyl scalars are all regular

in the interaction region. Cross polarization, however, does not remove the singularities formed on the null boundaries. In this paper we choose CPBS solution as a test ground, instead of BS, with various added test fields to justify the validity of a CH stability conjecture proposed previously by Helliwell and Konkowski (HK) [8,9]. Unlike the tedious perturbation analysis of both CX and Yurtsever the conjecture seems to be much economical in reaching a direct conclusion about the stability of a CH. This is our main motivation for considering the problem anew, for the case of untested solutions in CPW. In this paper we look at the spacetimes: a) single plane wave with added colliding test fields and b) colliding plane waves having non-singular interaction regions with test field added, Fig.1 illustrates these cases.

The terminology of singularities should be follwed from the classification presented by Ellis and Schmidt [10]. Singularities in maximal four dimensional spacetimes can be divided into three basic types: quasiregular (QR), scalar curvature (SC) and non-scalar curvature (NSC). The CH stability conjecture due to HK is defined as follows.

**Conjecture 1** *For all maximally extended spacetimes with CH, the back-reaction due to a field (whose test stress-energy tensor is $T_{\mu\nu}$) will affect the horizon in one of the following manners. a)If $T^{\mu}_{\mu}$, $T_{\mu\nu}T^{\mu\nu}$ and any null dust density $\rho$ are finite, and if the stress energy tensor $T_{ab}$ in all parallel propagated orthonormal (PPON) frames is finite, then the CH remains non-singular. b)If $T^{\mu}_{\mu}$, $T_{\mu\nu}T^{\mu\nu}$ and any null dust density $\rho$ are finite, but $T_{ab}$ diverges in some PPON frames, then an NSC singularity will be formed at the CH. c)If $T^{\mu}_{\mu}$, $T_{\mu\nu}T^{\mu\nu}$ and any null dust density $\rho$ diverges, then an SC singularity will be formed at the CH.*

Expressed otherwise, the conjecture suggests to put a test field into the background geometry and study the reaction it will experience. If certain scalars diverge then in an exact back-reaction solution the field will respond with

an infinite strength to the geometry (i.e action versus reaction). Such an infinite back-reaction will render a CH unstable and convert it into a scalar singularity.

The paper is organized as follows. In section II, we review the CPBS solution and the correct nature of the singularity structure is presented in Appendix A. Section III, deals with geodesics and test em and scalar field analyses. In section IV, we present an exact back reaction calculation for the collision of cross polarized em field coupled with scalar field. The derivation of Weyl and Maxwell scalars are given in Appendix B. The insertion of test null dusts to the background CPBS spacetime and its exact back reaction solution is studied in section V. Appendix C is devoted for the properties of this solution. The paper is concluded with a discussion in section VI.

## 2 The Cross-Polarized BS (CPBS) Metric

The metric that describes collision of em waves with the cross polarization was found to be [7]

$$ds^2 = F\left(\frac{d\tau^2}{\Delta} - \frac{d\sigma^2}{\delta}\right) - \Delta F dy^2 - \frac{\delta}{\Delta}(dx - q\tau dy)^2 \tag{1}$$

In this representation of the metric our notations are

$$\begin{aligned} \tau &= \sin(au + bv) \\ \sigma &= \sin(au - bv) \\ \Delta &= 1 - \tau^2 \\ \delta &= 1 - \sigma^2 \\ 2F &= \sqrt{1+q^2}(1+\sigma^2) + 1 - \sigma^2 \end{aligned} \tag{2}$$

in which $0 \leq q \leq 1$ is a constant measuring the second polarization, $(a, b)$ are constant of energy and $(u, v)$ stand for the usual null coordinates. It can be seen easily that for $q = 0$ the metric reduces to BS. Unlike the BS metric ,however, this is conformally non-flat for $(u > 0, v > 0)$, where the conformal curvature is generated by the cross polarization. As a matter of fact this solution is a minimal extension of the BS metric. A completely different generalization of the BS solution with second polarization was given by CX [11]. Their solution, however, employs an Ehlers transformation and involves two essential parameters which is therefore different from ours. Both solutions form CH in the interaction region. Our result drown out in this paper, namely, that the horizon is unstable against added sources can also be shown to apply to the CX metric as well. As it was shown before the interaction region ($i.e\, u > 0, v > 0$) of the above metric is of type-D without scalar curvature singularities. We wish to check now the possible singularities of metric (1). The single component of the Weyl scalar suffices to serve our purpose. We find that the real part of the Weyl scalar $\Psi_4$ is given by

$$\begin{aligned} Re\Psi_4 &= -\left(\frac{a\delta(u)\tan(bv)}{2(\Sigma^2 + q^2\sin^2(bv))}\right)\Bigg[\cos^4(\alpha/2) + \frac{q^2}{4}(5\sin^2 bv - 3) \\ &\quad - \cos(2bv)\sin^2(bv)\sin^4(\alpha/2)\Bigg] \end{aligned} \tag{3}$$

where we have used the abbreviations

$$\begin{aligned} \Sigma &= \cos^2(\alpha/2) + \sin^2(bv)\sin^2(\alpha/2) \\ q &= \sin\alpha \end{aligned} \tag{4}$$

As $q \rightarrow 0\,(or \alpha \rightarrow 0)$ we obtain

$$Re\Psi_4 = \Psi_4 = -\frac{a}{2}\delta(u)\tan(bv) \tag{5}$$

which reduces to the singularity form of the BS spacetime given by $u = 0, bv = \pi/2$. We see that the same singularity remains unaffected by the

introduction of the cross polarization. A similar calculation for $Re\Psi_0$ gives the symmetrical singular hypersurface sitting on $v = 0, au = \pi/2$. Now in order to explore the true nature of the singularity we concentrate our account on the incoming region II ($u > 0, v < 0$). The metric in this region is expressed in the form

$$ds^2 = 2e^{-M}dudv - e^{-U}[(e^V dx^2 + e^{-V}dy^2)\cosh W - 2\sinh W\, dxdy] \qquad (6)$$

where

$$\begin{aligned} e^{-M} &= 2F = 1 + \sqrt{1+q^2} + \left(\sqrt{1+q^2} - 1\right)\sin^2(au) \\ e^V \cosh W &= \frac{1}{F} \\ e^{-V}\cosh W &= F + \frac{q^2 \sin^2(au)}{F} \end{aligned}$$

We observe that for $q \neq 0$, $F(u)$ is a bounded positive definite function which suggests that no additional singularities arise except the one occuring already in the BS case, namely at $au = \pi/2, v = 0$. To justify this we have calculated all Riemann components in local and PPON frames (see Appendix A ). It is found that all Riemann tensor components vanish as $au \to \pi/2$. In the PPON frame, however, they are all finite and according to the classification scheme of Ellis and Schmidt such a singularity is called a quasiregular (QR) singularity. This is said to be the mildest type among all types of singularities. To check whether the QR is stable or not we consider generic test fields added to such a background geometry and study the effects. This we will do in the follwing sections.

## 3 Geodesics Behaviour, Test em and Test Scalar Fields

We are interested in the stability of QR singularities that are developed at $au = \pi/2$ in region II and $bv = \pi/2$ in region III. To investigate their stability we will express geodesics and behaviour of test em and scalar fields by calculating stress - energy tensor in local and PPON frames.

Our discussion on geodesics will be restricted in Region II only. We shall consider the geodesics that originate at the wave front and move toward the quasiregular singularity. Solution of geodesics equations in region II can be obtained by geodesics Lagrangian method and using $u$ as a parameter. The results are

$$\begin{aligned}
x - x_0 &= -\frac{2P_{x_0}\left[1+2q^2\right]}{a}\tan(au) + \frac{3P_{x_0}\left[5q^2+2-2\sqrt{1+q^2}\right]u}{4} \\
&\quad -\frac{P_{x_0}\left[\sqrt{1+q^2}-1\right]^2}{8a}\sin(2au) - \frac{2P_{y_0}q}{a\cos(au)} \\
y - y_0 &= -\frac{2qP_{x_0}}{a\cos(au)} - \frac{2P_{y_0}\tan(au)}{a} \\
v - v_0 &= \frac{\tan(au)}{a}\left[P_{x_0}^2(1+2q^2)+P_{y_0}^2\right] + u\left[\frac{\epsilon}{4}(1+3\sqrt{1+q^2})\right. \\
&\quad \left. -\frac{3P_{x_0}^2}{8}(5q^2+2-2\sqrt{1+q^2})\right] + \frac{2P_{x_0}P_{y_0}q}{a\cos(au)} \\
&\quad +\frac{(\sqrt{1+q^2}-1)\sin(2au)}{8a}\left[\frac{P_{x_0}^2}{2}(\sqrt{1+q^2}-1)-\epsilon\right]
\end{aligned} \tag{7}$$

where $\epsilon = 0$ for null and $\epsilon = 1$ for time like geodesics and $x_0, y_0, v_0, P_{x_0}$ and $P_{y_0}$ are constants of integration. It can be checked easily that for $q = 0$ our geodesics agree with those of the region II of the BS metric [8]. It is clear to see that if either $P_{x_0}$ or $P_{y_0}$ is nonzero then $v$ becomes positive for $u < \pi/2a$, and particles can pass from region II to the region IV. Geodesics that remain

in region II are

$$
\begin{aligned}
x &= x_0 \\
y &= y_0 \\
v &= v_0 + \frac{\pi\epsilon}{8a}\left(1 + 3\sqrt{1+q^2}\right)
\end{aligned}
\tag{8}
$$

where $v_0 < -\frac{\pi\epsilon}{8a}\left(1+3\sqrt{1+q^2}\right)$. The effect of cross polarization is that more geodesics remains in region II relative to the parallel polarization case. On physical grounds this result could be anticipated because cross polarization behaves like rotation which creates a pushing out effect in the non - inertial frames.

## 3.1 Test em field:

To test the stability of quasiregular singularity, let us consider a test em field whose vector potential is choosen appropriately as in [9] to be

$$
A_\mu(v) = (0, 0, f_1(v), f_2(v)) \tag{9}
$$

with arbitrary functions $f_1(v)$ and $f_2(v)$. The only nonzero energy - momentum for this test em field is

$$
4\pi T_{vv} = -\frac{1}{F\cos^2(au)}\left[\left(F^2 + q^2\sin^2(au)\right)f_1'^2 + f_2'^2 + 2q\sin(au)f_1'f_2'\right] \tag{10}
$$

in which a prime denotes derivative with respect to $v$. Both of scalars $T^\mu_\mu$ and $T_{\mu\nu}T^{\mu\nu}$ vanish, predicting that QR singularities are not transformed into a SCS. In the PPON frame.

$$
\begin{aligned}
e_{(0)\mu} &= (F, 1, 0, 0) \\
e_{(1)\mu} &= (-F, 1, 0, 0) \\
e_{(2)\mu} &= \left(0, 0, \cos(au)e^{V/2}cosh\frac{W}{2}, -\cos(au)e^{-V/2}sinh\frac{W}{2}\right) \\
e_{(3)\mu} &= \left(0, 0, -\cos(au)e^{V/2}sinh\frac{W}{2}, \cos(au)e^{-V/2}cosh\frac{W}{2}\right)
\end{aligned}
\tag{11}
$$

We find that $T_{(mn)}$ are given in terms of $T_{vv}$ by

$$4T_{(mn)} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} T_{vv}$$

for$(m, n = 0, 1)$ and $T_{(mn)} = 0$, otherwise. The divergence of this quantity predicts the occurence of NSCS and therefore QR singularity must be unstable.

The stability conjecture therefore correctly finds that these QR singularities are unstable. However, the same stability conjecture does not find correctly the nature of the singularity. As we have discussed in section II, the interior of the interaction region has no SCS. The only SCS is on the null boundaries. Clarke and Hayward have analysed these singular points for a collinear BS spacetime and found that the singularity nature of surfaces $(u = 0, v = \pi/2b)$ and $(v = 0, u = \pi/2a)$ are QR. This observation can also be used in the cross polarized version of BS spacetime , because the order of diverging terms in $\Psi_0$ and $\Psi_4$ are the same.

## 3.2 Test Scalar Field:

The QR singularity structure formed in the incoming region of BS problem remains unchanged in the case of cross polarized version of the same problem. Let us now consider the stability of these QR singularities by imposing a test scalar field in region II which is the one of the incoming region bounded by the QR singularity. The massless scalar field equation is given by

$$\partial_\mu \left( g^{\mu\nu} \sqrt{g} \phi_{,\nu} \right) = 0 \tag{12}$$

where we consider $x, y$ independent scalar waves so that a particular solution to this equation is obtained as in the ref()

$$\phi(u, v) = g(u) + \sec(au) f(v) \tag{13}$$

where $g(u)$ and $f(v)$ are arbitrary functions. The stress energy tensor is given by

$$T_{\mu\nu} = \frac{1}{4\pi}\left(\phi_\mu\phi_\nu - \frac{1}{2}g_{\mu\nu}\phi_\alpha\phi^\alpha\right) \tag{14}$$

The corresponding non-zero stress-energy tensors for the test scalar wave is obtained by taking $g(u) = 0$ as,

$$\begin{aligned}
T_{uu} &= \frac{a^2\sec^2(au)\tan^2(au)f^2(v)}{4\pi} \\
T_{vv} &= \frac{\sec^2(au)f'^2(v)}{4\pi} \\
T_{xx} &= \frac{a\tan(au)f'(v)f(v)}{8\pi F^2} \\
T_{yy} &= \frac{a\tan(au)\left[F^2 + q^2\sin^2(au)\right]f(v)f'(v)}{8\pi F^2} \\
T_{xy} &= T_{yx} = \frac{aq\sin(au)\tan(au)f(v)f'(v)}{8\pi F^2}
\end{aligned} \tag{15}$$

It is observed that each component diverges as the QR singularity $au \rightarrow \pi/2$ is approached.

Next we consider the stress energy tensor in a PPON frame. Such frame vectors are given in equation (11). The stress-energy tensor is

$$T_{(ab)} = e^\mu_{(a)}e^\nu_{(b)}T_{\mu\nu} \tag{16}$$

The nonzero components are;

$$\begin{aligned}
T_{00} &= T_{11} = \left(\frac{\sec^2(au)}{16\pi}\right)\left[\frac{a^2\tan^2(au)f^2}{F^2} + f'^2(v)\right] \\
T_{01} &= T_{10} = \left(\frac{\sec^2(au)}{16\pi}\right)\left[\frac{a^2\tan^2(au)f^2}{F^2} - f'^2(v)\right] \\
T_{22} &= T_{33} = \left(\frac{a\sec^2(au)\tan(au)f(v)f'(v)}{8\pi F^3}\right)\left[F^2 + 2q^2\sin^2(au)\right] \\
T_{32} &= T_{23} = \frac{aq\sec^3(au)\sin^2(au)f'(v)f(v)\sqrt{F^2 + q^2\sin^2(au)}}{4\pi F^3}
\end{aligned} \tag{17}$$

These components also diverge as the singularity $au \to \pi/2$ is approached.

By the conjecture, this indicates that the QR singularity will be transformed into a curvature singularity. Finally we calculate the scalar $T_{\mu\nu}T^{\mu\nu}$.

$$
\begin{aligned}
T_{\mu\nu}T^{\mu\nu} &= \frac{a^2 \sec^4(au)\tan^2(au) f^2(v) f'^2(v)}{64\pi^2 F^6} \Big\{ 2F^4 + 2q^4 \sin^4(au) \\
&\quad + \Big(F^2 + q^2 \sin^2(au)\Big) \Big[ 13 q^2 \sin^2(au) + F^2 + 1 \Big] \Big\}
\end{aligned}
$$

which also diverges as $au \to \pi/2$. From these analyses we conclude that the curvature singularity formed will be an SCS.

Hence, the conjecture predicts that the QR singularities of cross polarized version of BS spacetime are unstable. It is predicted that the QR singularities will be converted to scalar curvature singularities if generic waves are added. The similar results have also been obtained by HK for the BS spacetime. HK was unable to compare the validity of the conjecture by an exact back-reaction solution. In the next section we present an explicit example that represents cross-polarized em field coupled with scalar field.

# 4 Testing The Conjecture For a Class of Einstein-Maxwell-Scalar (EMS) Solutions.

In the former sections, we applied HK stability conjecture to test the stability of QR singularities in the incoming region of CPBS spacetime. According to the conjecture these mild singularities are unstable. In order to see the validity of the conjecture we introduce this new solution.

Let the metric be;

$$ds^2 = 2e^{-M}dudv - e^{-U}[(e^V dx^2 + e^{-V}dy^2)\cosh W - 2\sinh W\, dxdy] \quad (18)$$

The new solution is obtained from the electrovacuum solution. The EMS solution is generated in the following manner. The Lagrangian density of the system is defined by

$$\begin{aligned} L &= e^{-U}\left(M_uU_v + M_vU_u + U_uU_v - W_uW_v - V_uV_vcosh^2W - 4\phi_u\phi_v\right) \\ &\quad -2k\left[\left(B_uB_ve^V + A_uA_ve^{-V}\right)coshW\right. \\ &\qquad\qquad \left. + \left(A_uB_v + A_vB_u\right)sinhW\right] \end{aligned} \quad (19)$$

which correctly generate all EMS field equations from a variational principle. Here $\phi$ is the scalar field and we define the em potential one-form (with coupling constant $k$) by

$$\tilde{A} = \tilde{A}_\mu dx^\mu = Adx + Bdy \quad (20)$$

where $A$ and $B$ are the components in the Killing directions. Variation with respect to $U, V, M, W, A, B$ and $\phi$ yields the following EMS equations.

$$U_{uv} = U_uU_v \quad (21)$$

$$2M_{uv} = -U_uU_v + W_uW_v + V_uV_vcosh^2W + 4\phi_u\phi_v \quad (22)$$

$$\begin{aligned} 2V_{uv} &= U_vV_u + U_uV_v - 2\left(V_uW_v + V_vW_u\right)tanhW \\ &\quad -2ksechW\left(\bar{\Phi_0}\Phi_2 + \bar{\Phi_2}\Phi_0\right) \end{aligned} \quad (23)$$

$$\begin{aligned} 2W_{uv} &= U_vW_u + U_uW_v + 2V_uV_vcoshW\,sinhW \\ &\quad +2ki\left(\bar{\Phi_0}\Phi_2 - \bar{\Phi_2}\Phi_0\right) \end{aligned} \quad (24)$$

$$2\phi_{uv} = U_v\phi_u + U_u\phi_v \quad (25)$$

$$\begin{aligned} 2A_{uv} &= V_vA_u + V_uA_v - tanhW\left(W_vA_u + W_uA_v\right) \\ &\quad -e^V\left[2A_{uv}tanhW + W_uA_v + W_vA_u\right] \end{aligned} \quad (26)$$

$$\begin{aligned}2B_{uv} &= -V_vB_u - V_uB_v - tanhW\left(W_vB_u + W_uB_v\right)\\&\quad -e^V\left[2B_{uv}tanhW + W_uB_v + W_vB_u\right]\end{aligned} \tag{27}$$

where $\Phi_0$ and $\Phi_2$ are the Newmann-Penrose spinors for em plane waves given as follows

$$\begin{aligned}\Phi_2 &= \frac{e^{U/2}}{\sqrt{2}}\left[e^{-V/2}\left(isinh\frac{W}{2} - cosh\frac{W}{2}\right)A_u\right.\\&\quad\left.+e^{V/2}\left(icosh\frac{W}{2} - sinh\frac{W}{2}\right)B_u\right]\\\Phi_0 &= \frac{e^{U/2}}{\sqrt{2}}\left[e^{-V/2}\left(isinh\frac{W}{2} + cosh\frac{W}{2}\right)A_v\right.\\&\quad\left.+e^{V/2}\left(icosh\frac{W}{2} + sinh\frac{W}{2}\right)B_v\right]\end{aligned} \tag{28}$$

The remaining two equations which do not follow from the variations, namely

$$\begin{aligned}2U_{uu} - U_u^2 + 2M_uU_u &= W_u^2 + V_u^2cosh^2W + 4\phi_u^2 + 4k\Phi_2\bar{\Phi}_2\\2U_{vv} - U_v^2 + 2M_vU_v &= W_v^2 + V_v^2cosh^2W + 4\phi_v^2 + 4k\Phi_0\bar{\Phi}_0\end{aligned} \tag{29}$$

are automatically satisfied by virtue of integrability equations.
The metric function $M$ can be shifted now in accordance with

$$M = \tilde{M} + \Gamma \tag{30}$$

where

$$\begin{aligned}\Gamma_uU_u &= 2\phi_u^2\\\Gamma_vU_v &= 2\phi_v^2\end{aligned} \tag{31}$$

and $U, V, \tilde{M}, W$ satisfy the electrovacuum EM equations. Integrability condition for the equation (31) imposes the constraint,

$$\left(\phi_uU_v - \phi_vU_u\right)\left[2\phi_{uv} - U_v\phi_u - U_u\phi_v\right] = 0 \tag{32}$$

from which we can generate a large class of EMS solution. As an example, for any $\phi$ satisfying the massless scalar field equation the corresponding $\Gamma$ is obtained from

$$\Gamma = 2\int \frac{\phi_u^2}{U_u}\,du + 2\int \frac{\phi_v^2}{U_v}\,dv \tag{33}$$

The only effect of coupling a scalar field to the cross polarized em wave is to alter the metric into the form,

$$ds^2 = Fe^{-\Gamma}\left(\frac{d\tau^2}{\Delta} - \frac{d\sigma^2}{\delta}\right) - \Delta F dy^2 - \frac{\delta}{F}\left(dx - q\tau dy\right)^2 \tag{34}$$

Here $F, \Delta, \delta$ and $q$ represents the solution of electrovacuum equations and $\Gamma$ is the function that derives from the presence of the scalar field.

It can be easily seen that for $q = 0$ our solution represents pure em BS solution coupled with scalar field. It constitutes therefore an exact back reaction solution to the test scalar field solution in the BS spacetime considered by HK (). It is clear to see that the Weyl scalars are nonzero and SCS is forming on the surface $au + bv = \pi/2$. This is in aggrement with the requirement of stability conjecture introduced by HK. For $q \neq 0$ the obtained solution forms the exact back reaction solution of the test scalar field solution in the CPBS spacetime. In Appendix B, we present the Weyl and Maxwell scalars explicitly.

From the explicit solutions we note that, the Coulomb component $\Psi_2$ remains regular but $\Psi_4$ and $\Psi_0$ are singular when $\tau = 1$ or $\sigma = 1$. This indicates that the singularity structure of the new solution is a typical SCS. This result is in complete agreement with the stability conjecture.

# 5 Oppositely Moving Null Dusts In CPBS Spacetime

**A)** Let us assume first null test dusts moving in the CPBS background. For simplicity we consider two different cases the $x =$ constant and $y =$ constant projections of the spacetime. We have in the first case

$$ds^2 = \frac{e^{-M}}{2ab}\left(dt^2 - dz^2\right) - e^{-U-V} coshW \, dy^2 \tag{35}$$

where we have used the coordinates $(t, z)$ according to

$$\begin{aligned} t &= au + bv \\ z &= au - bv \end{aligned} \tag{36}$$

The energy-momentum tensor for two oppositely moving null dusts can be chosen as

$$T_{\mu\nu} = \rho_l l_\mu l_\nu + \rho_n n_\mu n_\nu \tag{37}$$

where $\rho_l$ and $\rho_n$ are the finite energy densities of the dusts. The null propagation directions $l_\mu$ and $n_\nu$ are

$$\begin{aligned} l_\mu &= (a_0, 0, a_2, a_3) \\ n_\mu &= (-a_0, 0, a_2, a_3) \end{aligned}$$

with

$$a_2 = k_2 = \text{constant} \qquad\qquad a_3 = \frac{k_1}{2ab} = \text{constant}$$

$$a_0 = \frac{1}{2ab}\left(k_1^2 + \frac{2abk_2^2}{coshW}\, e^{U+V-M}\right)^{1/2}$$

We observe from (1) that

$$\frac{e^{U+V}}{coshW} = \frac{F}{\Delta F^2 + \delta q^2 \tau^2} \tag{38}$$

which is finite for $q \neq 0$. The components of energy-momentum tensors in PPON frames are proportional to the expression (38). This proportionality makes all the components of energy-momentum tensors finite. In the limit as $q \rightarrow 0$ which reduces our line element to the BS this expression diverges on the horizon $(\tau = 1)$ . Trace of the energy-momentum is obviously zero therefore we must extract information from the scalar $T_{\mu\nu}T^{\mu\nu}$. One obtains,

$$T_{\mu\nu}T^{\mu\nu} = 2\rho_l\rho_n \left( \frac{k_1^2 e^M}{ab} + \frac{2k_2^2 e^{U+V}}{coshW} \right)^2 = \text{finite} \tag{39}$$

The projection on $y =$ constant, however is not as promising as the $x =$ constant case. Consider the reduced metric

$$ds^2 = \frac{e^{-M}}{2ab} \left( dt^2 - dz^2 \right) - e^{-U+V} coshW \, dx^2 \tag{40}$$

A similar analysis with the null vectors

$$\begin{aligned} l_\mu &= (a_0, a_1, 0, a_3) \\ n_\mu &= (-a_0, a_1, 0, a_3) \end{aligned}$$

with

$$a_1 = k_3 = \text{constant} \qquad\qquad a_3 = \frac{k_1}{2ab} = \text{constant}$$

$$a_0 = \frac{1}{2ab} \left( k_1^2 + \frac{2abk_3^2}{coshW} \, e^{U-V-M} \right)^{1/2}$$

yields the scalar

$$T_{\mu\nu}T^{\mu\nu} = 2\rho_l\rho_n \left( \frac{k_1^2 e^M}{ab} + \frac{2k_3^2 e^{U-V}}{coshW} \right)^2 \tag{41}$$

From the metric (1) we see that

$$\frac{e^{U-V}}{coshW} = \frac{F}{\delta} \tag{42}$$

which diverges on the horizon $\sigma = 1$. The scalar $T_{\mu\nu}T^{\mu\nu}$ constructed from the test dusts therefore diverges. The PPON components of the energy momentum tensors are also calculated and it is found that all of the components are proportional to the expression (42). This indicates that the components of energy-momentum tensor diverges as the singularity is approached. When we consider the HK stability conjecture an exact back reaction solution must give a singular solution. We present now an exact back reaction solution of two colliding null shells in the interaction region of the CPBS spacetime.

**B)** The metric given by

$$ds^2 = \frac{1}{\phi^2}\left(2dudv - dx^2 - dy^2\right) \tag{43}$$

where $\phi = 1 + \alpha u\theta(u) + \beta v\theta(v)$ with $(\alpha, \beta)$ positive constants was considered by Wang [12] to represent collision of two null shells (or impulsive dusts). The interaction region is transformable to the de Sitter cosmological spacetime. In other words the tail of two crossing null shells is energetically equivalent to the de Sitter space. It can be shown also that the choice of the conformal factor $\phi = 1 + \alpha u\theta(u) - \beta v\theta(v)$, with $(\alpha, \beta)$ positive constants becomes isomorphic to the anti - de Sitter space.

The combined metric of CPBS and colliding shells can be represented by

$$ds^2 = \frac{1}{\phi^2}\, ds^2_{CPBS} \tag{44}$$

This amounts to the substitutions

$$\begin{aligned} M &= M_0 + 2\ln\phi \\ U &= U_0 + 2\ln\phi \\ V &= V_0 \\ W &= W_0 \end{aligned} \tag{45}$$

where $(M_0, U_0, V_0, W_0)$ correspond to the metric functions of the CPBS solution. Under this substitutions the scale invariant Weyl scalars remain invariant ( or at most multiplied by a conformal factor ) because $M - U = M_0 - U_0$ is the combination that arise in those scalars. The scalar curvature, however, which was zero in the case of CPBS now arises as nonzero and becomes divergent as we approach the horizon. Appendix C gives the scalar quantities $\Lambda, \Phi_{11}, \Phi_{00}, \Phi_{22}$ and $\Phi_{02}$. Thus the exact back reaction solution is a singular one as predicted by the conjecture. It is further seen that choosing $\beta = 0$, which removes one of the shells leaves us with a single shell propagating in the $v$- direction. From the scalars we see that even a single shell gives rise to a divergent back reaction by the spacetime. The horizon, in effect, is unstable and transforms into a singularity in the presence of two colliding, or even a single propagating null shell. Let us note as an alternative interpretation that the metric (43) may be considered as a colliding em waves in a de Sitter background. Collision of em waves creates an unstable horizon in the de Sitter space which is otherwise regular for $u > 0$ and $v > 0$.

# 6 Discussion

In this paper we have analysed the stability of QR singularities in the CPBS spacetime. Three types of test fields are used to probe the stability. First we have applied test em field to the background CPBS spacetime. From the analyses we observed that the QR singularity in the incoming region becomes unstable according to the conjecture, and it is transformed into NSC singularity. This is the prediction of the conjecture. However, the exact back-reaction solution shoes that beside the true singularities on the null boundaries $u = 0, v = \frac{\pi}{2b}$ and $v = 0, u = \frac{\pi}{2a}$. There are quasiregular singularities in the incoming regions. The interior of interaction region is singularity

free and the hypersurface $au + bv = \pi/2$ is a Killing-Cauchy horizon. As it was pointed out by HK in the case of colliding em step waves, conjecture fails to predict the correct nature of the singularity in the interaction region. We have also discovered the same behaviour for the cross polarized version of colliding em step waves. The addition of cross polarization does not alter the existing property.

Secondly we have applied test scalar field to the background CPBS spacetime. The effect of scalar field on the QR singularity is stronger than the effect of em test field case. We have obtained that the QR singularity is unstable and transformes into a SCS. To check the validity of the conjecture , we have constructed a new solution constituting an exact back reaction solution to the test scalar field in the CPBS spacetime. The solution represents the collision of cross polarized em field coupled to a scalar field. An examination of Weyl and Maxwell scalars shows that $\Psi_0, \Psi_4, \Phi_{00}$ and $\Phi_{22}$ diverge as the singularity is approached and unlike the test em field case the conjecture predicts the nature of singularity formed correctly.

Finally, we have introduced test null dusts into the interaction region of CPBS spacetime. The conjecture predicts that the CH are unstable and transforms into SCS. This result is compared with the exact back-reaction solution and observed that the conjecture works.

# Appendix A:

# Riemann Components for Region II

To determine the type of singularity in the incoming region of CPBS spacetime, the Riemann tensor in local and in PPON frame must be evaluated. Non-zero Riemann tensors in local coordinates are found as follows.

$$\begin{aligned}
-R_{uxux} &= e^{V-U}\left[\Phi_{22}coshW + (Im\Psi_4)sinhW + Re\Psi_4\right] \\
-R_{uyuy} &= e^{-U-V}\left[\Phi_{22}coshW + (Im\Psi_4)sinhW - Re\Psi_4\right] \\
R_{uxuy} &= e^{-U}\left[\Phi_{22}sinhW + (Im\Psi_4)coshW\right] \qquad (46)
\end{aligned}$$

where

$$\begin{aligned}
Re\Psi_4 &= -\frac{1}{2}\left[\left(V_{uu} - V_uU_u + M_uV_u\right)coshW + 2V_uW_usinhW\right] \\
Im\Psi_4 &= -\frac{i}{2}\left(W_{uu} - U_uW_u + M_uW_u - V_u^2coshW sinhW\right) \\
\Phi_{22} &= \frac{1}{4}\left(2U_{uu} - U_u^2 - W_u^2 - V_u^2cosh^2W\right)
\end{aligned}$$

Note that in region II the Weyl scalar $\Psi_4 = 0$, therefore only $\Phi_{22}$ exists. It is clear that $e^{-U} = 0$ in the limit $au \to \pi/2$, so that all of the components vanish

$$R_{uxuy} = R_{uyuy} = R_{uxux} = 0$$

To find the Riemann tensor in a PPON frame, we define the following PPON frame vectors;

$$\begin{aligned}
e^{\mu}_{(0)} &= \left(\frac{1}{2F}, \frac{1}{2}, 0, 0\right) \\
e^{\mu}_{(1)} &= \left(\frac{1}{2F}, -\frac{1}{2}, 0, 0\right) \\
e^{\mu}_{(2)} &= \left(0, 0, -e^{\frac{U-V}{2}}cosh\frac{W}{2}, -e^{\frac{U+V}{2}}sinh\frac{W}{2}\right) \\
e^{\mu}_{(3)} &= \left(0, 0, -e^{\frac{U-V}{2}}sinh\frac{W}{2}, -e^{\frac{U+V}{2}}cosh\frac{W}{2}\right)
\end{aligned}$$

In this frame the non-zero components of the Riemann tensors are:

$$
\begin{aligned}
R_{0202} &= -\frac{1}{4F^2}\left(\Phi_{22} + Re\Psi_4\right) \\
R_{0303} &= -\frac{1}{4F^2}\left(\Phi_{22} - Re\Psi_4\right) \\
R_{1313} &= -\frac{1}{4F^2}\left(\Phi_{22} - Re\Psi_4\right) \\
R_{1212} &= -\frac{1}{4F^2}\left(\Phi_{22} + Re\Psi_4\right) \\
R_{0302} &= R_{1312} = \frac{1}{4F^2} Im\Psi_4
\end{aligned}
\tag{47}
$$

In the limit of $au \to \pi/2$, we have the following results

$$
\begin{aligned}
R_{0202} &= R_{1212} = R_{0303} = R_{1313} = -\frac{a^2}{(1+q^2)^{3/2}} \\
R_{0302} &= R_{1312} = 0
\end{aligned}
$$

which are all finite. This indicates that the apparent singularity in region II (one of the incoming region) is a quasiregular singularity.

# Appendix B:

# Properties of the EMS Solution

In order to calculate the Weyl and Maxwell scalars we make use of the CX line element

$$
ds^2 = U^2\left(d\psi^2 - d\theta^2\right) - \frac{\sin\phi\sin\theta}{1-\epsilon\bar{\epsilon}}\left|(1-\epsilon)dx + i(1+\epsilon)dy\right|^2 \tag{48}
$$

where

$$
\begin{aligned}
U^2 &= Fe^{-\Gamma} \qquad\qquad \epsilon = \frac{Z-1}{Z+1} \\
Z &= \sin\theta\left(\Sigma\sin\psi - i\sin\alpha\sin\theta\cos\psi\right)^{-1}
\end{aligned}
$$

$\Sigma$ is given in equation (4) and we have chosen $a = 1 = b$, such that the new coordinates $(\theta, \psi)$ are related to $(\tau, \sigma)$ by

$$\begin{aligned}\tau &= \cos\psi \\ \sigma &= \cos\theta\end{aligned}$$

The Weyl and Maxwell scalars are found as

$$\Psi_2 = Re^{\Gamma} \tag{49}$$

$$\begin{aligned}\Psi_4 = & -e^{\Gamma+i\lambda}\Bigg[3R + \frac{1}{4F\Sigma\sin\theta\sin\psi}\big(\Sigma\sin(\theta+\psi) - \Sigma_\theta\sin\psi\sin\theta \\ & +i\sin\alpha\sin^2\theta\sin\psi\big)\left(\Gamma_\psi - \Gamma_\theta\right)\Bigg]\end{aligned} \tag{50}$$

$$\begin{aligned}\Psi_0 = & -e^{\Gamma-i\lambda}\Bigg[3R + \frac{1}{4F\Sigma\sin\theta\sin\psi}\big(\Sigma\sin(\psi-\theta) - \Sigma_\theta\sin\psi\sin\theta \\ & +i\sin\alpha\sin^2\theta\sin\psi\big)\left(\Gamma_\psi + \Gamma_\theta\right)\Bigg]\end{aligned} \tag{51}$$

$$2\Phi_{00} = e^{\Gamma}\left[\frac{\cos\alpha}{\Sigma^2} - \frac{\sin(\psi+\theta)(\Gamma_\theta + \Gamma_\psi)}{2F\sin\psi\sin\theta}\right] \tag{52}$$

$$2\Phi_{22} = e^{\Gamma}\left[\frac{\cos\alpha}{\Sigma^2} - \frac{\sin(\theta-\psi)(\Gamma_\theta - \Gamma_\psi)}{2F\sin\psi\sin\theta}\right] \tag{53}$$

$$-2\Phi_{02} = e^{\Gamma+i\lambda}\frac{\cos\alpha}{\Sigma^2} \tag{54}$$

where

$$\begin{aligned}R &= \left(\frac{\sin(\alpha/2)}{2\Sigma}\right)\left[\frac{\sin(\alpha/2) - i\cos\theta\cos(\alpha/2)}{\cos(\alpha/2) + i\cos\theta\sin(\alpha/2)}\right] \\ e^{i\lambda} &= \frac{\sin\theta + \Sigma\sin\psi + i\sin\psi\sin\theta\cos\psi}{\sin\theta + \Sigma\sin\psi - i\sin\psi\sin\theta\cos\psi}\end{aligned}$$

# Appendix: C

# The Weyl and Maxwell Scalars

The non-zero Weyl and Maxwell scalars for the collision of null shells in the background of CPBS spacetime are found as follows.

$$\begin{aligned}
4\phi e^{-M}\Phi_{11} &= [(a\beta+\alpha b)\tan(au+bv) \\
&\quad +(a\beta-\alpha b)\tan(au-bv)]\,\theta(u)\theta(v) \qquad (55)
\end{aligned}$$

$$\begin{aligned}
4\phi e^{-M}\Lambda &= \left[(a\beta+\alpha b)\tan(au+bv)+(a\beta-\alpha b)\tan(au-bv)\right. \\
&\quad \left.+\frac{4\alpha\beta}{\phi}\right]\theta(u)\theta(v) \qquad (56)
\end{aligned}$$

$$\begin{aligned}
\Phi_{22} &= (\Phi_{22})_{CPBS}+\left(\frac{\alpha e^{M}}{\phi}\right)\left[\delta(u)\right. \\
&\quad \left.-\theta(u)\left(a\Pi+\frac{u}{(1-u^2)(1-v^2)}\right)\right] \qquad (57)
\end{aligned}$$

$$\begin{aligned}
\Phi_{00} &= (\Phi_{00})_{CPBS}+\left(\frac{\beta e^{M}}{\phi}\right)\left[\delta(v)\right. \\
&\quad \left.+\theta(v)\left(b\Pi-\frac{v}{(1-u^2)(1-v^2)}\right)\right] \qquad (58)
\end{aligned}$$

$$\begin{aligned}
\Phi_{02} &= (\Phi_{02})_{CPBS}+\left(\frac{e^{M}}{4FY\phi}\right)\left[\frac{1}{F}\left(\alpha Q\theta(u)+\beta P\theta(v)\right)\right. \\
&\quad \left.+iq\left(\alpha L\theta(u)+\beta K\theta(v)\right)\right] \qquad (59)
\end{aligned}$$

where

$$
\begin{aligned}
\phi &= 1 + \alpha u \theta(u) + \beta v \theta(v) \\
Q &= b \Big[ 2q^2 \sin(au+bv)\cos(au-bv) - F^2 \left( \tan(au-bv) + \tan(au+bv) \right) \\
&\quad -2F \cos(au-bv)\sin(au-bv) \left( \sqrt{1+q^2} - 1 \right) \Big] \\
P &= a \Big[ 2q^2 \sin(au+bv)\cos(au-bv) + F^2 \left( \tan(au-bv) - \tan(au+bv) \right) \\
&\quad +2F \cos(au-bv)\sin(au-bv) \left( \sqrt{1+q^2} - 1 \right) \Big] \\
Y &= \left( 1 + \frac{q^2}{F^2} \tan(au+bv)\sin(au+bv)\cos(au-bv) \right)^{1/2} \\
K &= \frac{a}{\sqrt{\cos(au+bv)\cos(au-bv)}} \left[ \frac{\cos(au-bv)}{\cos(au+bv)} + \sin 2au \right. \\
&\quad \left. - \frac{2\left(\sqrt{1+q^2}-1\right)\sin(au+bv)\cos(au-bv)\tan(au-bv)}{F} \right] \\
L &= \frac{b}{\sqrt{\cos(au+bv)\cos(au-bv)}} \left[ \frac{\cos(au-bv)}{\cos(au+bv)} + \sin 2bv \right. \\
&\quad \left. + \frac{2\left(\sqrt{1+q^2}-1\right)\sin(au+bv)\cos(au-bv)\tan(au-bv)}{F} \right] \\
\Pi &= \frac{\left(\sqrt{1+q^2}-1\right)\sin(2au-2bv)}{\sqrt{1+q^2}+1+\left(\sqrt{1+q^2}-1\right)\sin^2(au-bv)}
\end{aligned}
$$

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# Figure caption

Fig.1(a): Single plane waves with added colliding test fields indicated by arrows. CH exists on the surface $u = 1$.
(b): Colliding plane wave spacetime with CH's in the incoming regions at $u = 1$ and $v = 1$. Test fields are added to test the stability of CH existing in region IV.

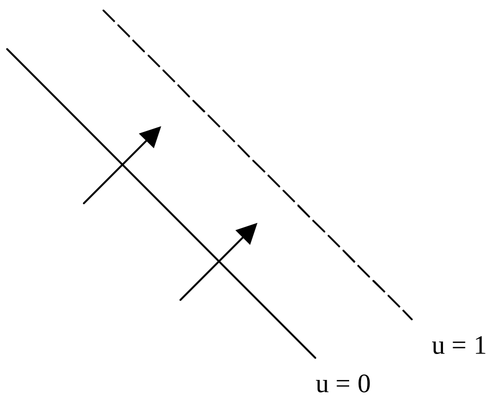


Fig. 1(a)

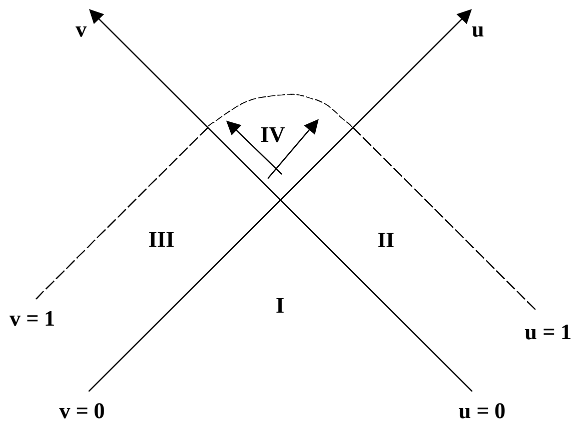


Fig.1(b)